CHAOS THEORY

Words like chaos, randomness and complexity are elevated from their ordinary connotation to a scientific level in the study of disorderly behaviour in physical, biological and chemical sciences as well as in different aspects of social sciences, economics, history and philosophy. It may be puzzling to many that such a wide spectrum of fields exhibits commonalities that compel intellectual but may be promiscuous dialogue. Such conversation and flow of information cannot be unidirectional and the discussion must involve contributions from all conceivable fields of learning. "There is nothing wrong in taking concepts developed in one science to see how they work in another if they yield an apt description." {R.N.Adams – The English Day (1988)}. Savants from different fields have evolved a new way of understanding the growth of complexity in nature. This new science, called chaos, offers a way of seeing order and pattern where formerly only the erratic was observed.

Our world is quite diverse in form and composition and a standard classification is not possible. A push here, a pull there may result in a major event; as against this, a major effort may yield very little. Cause and effect are not proportional and significant effect may be the result of a small cause or vice-versa. Mathematics describes this well by the brief statement: the world is nonlinear.

We are also bedevilled by chance in our daily living, even if we do not invest money on lottery tickets. Often we experience unexpected joy and at times we meet with unforeseen misfortune.

Thus in the dynamics of our daily lives we have to reckon with nonlinearity and stochasticity or chance, the two ingredient segments of complexity. In early days matters like complexity were debated by

natural philosophers who were also scientists. With the arrival of Newton (1642-1727) on the scene, determinism received impetus and the subsequent scenario is tellingly described by the phrase: Laplace's Demon: "We may regard the present state of the universe as the effect of the past and the cause of the future. An intellect which at any given moment knew all the forces that animate nature and the mutual positions of the beings that compose it, if this intellect were vast enough to submit the data to analysis, could condense into a single formula the movement of the greatest bodies of the universe and that of the lightest atom; for such an intellect nothing could be uncertain and the future just like the past, would be present before our eyes." {M.Kline; Mathematics in Western Culture (1953)}. While such deterministic belief has its attractive features, we have to reckon with the fact that our own personal experience does not support this line of thought. Newton related the science of his day to the creator but Laplace felt no such compulsion and believed that the universe functions rationally according to the laws of mechanics (= the grammar of physics) with no uncertainty, no chance, no choice, no freedom and no free will.

However, strict determinism is to be ruled out of consideration in science, since measurements are affected by the presence of the observer. The number of particles of a system is of the order of (27) times (10 raised to18) = 27 followed by eighteen zeros per cubic centimetre and their positions and momenta cannot be specified precisely, but only in a statistical way. The population of the world is of the order 53 followed by eight zeros and even for this figure, we cannot indicate the whereabouts of the individuals on their activities. Uncertainty is a fact of life.

A generic illustration will be helpful in relating the phenomenon of complexity and chaos. If (x) denotes the normalized population and (a) is

a parameter – not necessarily normalized, we have the discrete logistic equation

$$X_{n+1} = a\, x_n (1-x_n) \qquad (1)$$

Which exhibits the three important ingredient features of (i.) forcing – presence of the factor or parameter (a) in the equation, (ii) nonlinearity – the presence of the term X_n raised to power 2 and (iii) dissipation arising from the factor $(1-x_n)$. The equation (1) can be looked upon as a means of finding subsequent itinerants when an initial value is specified. It is thus a means of updating the value of X as the count (n) takes place. The equation possesses an unexpectedly complicated display of behaviour of the surrogate variable X, embracing a variety of problems in diverse fields like (a) population dynamics, (b) economics, (c) biological problems and (d) ecology. The parameter (a) in equation (1) has a crucial role in the understanding of chaos and for very valid reasons we restrict it to the range $o< \underline{a} \leq 4$. The range $o< \underline{a} \leq 1$ is important for the mathematical analysis of the problem and corresponds to the segment of orderly behaviour. The equation is a candidate for exhibition of disorderly behaviour or chaos when the parameter has the range $1\leq \underline{a} \leq 4$.

The dynamics of a complex system varies and we can experience this during travel when we cross different time zones and our normal biological rhythms lag or advance. Complex dynamical systems are neither completely deterministic nor completely random and they exhibit both characteristics. Different parts of complex systems are linked and affect one another in a synergetic manner.

We must distinguish between chaos and chaos theory. Chaos is a condition; chaos theory is an amalgam of methods used to scrutinize nonlinear, dissipative, deterministic problems that have randomness embedded in them. Chaos occurs in systems that are sensitive to initial

conditions. Even a large system may become chaotic if at some point a miniscule stimulus perturbs the system. Chaotic phenomena, wherever they may arise – (whether in mathematics or in nature or in life or in the brains) – are in the strict technical sense, apparent randomness. Consideration of such phenomena are in the context of how, apparently random phenomena are contain hidden orderliness and also how orderly phenomena become random. Consider the two symbols 'A' and 'B' and the eighteen digit strings of these symbols.

(i) B A AB ABBB AB AA BBA BAA; (ii) B BBBBBBBB AAAAAAAAA

Our intuitive perception is that the first displays disorder or chaos and the second is an orderly arrangement. Transition from (i.) to (ii) is a change from disorder to order and that from (ii) to (i.) is a change from order to disorder. Such transitions occur in natural systems when there is change in the initial conditions or a change in the control parameter in the model equation characterizing the dynamics of the system. The destiny of the system is determined by the initial state of the system and the value of the control parameter. Chaos theory is a scientific theory to tackle such questions in nonlinear dynamical systems, regardless of the entities of the system. It provides insights to nonlinear phenomena that involve random aspects. It is a general and powerful paradigm for the study of complex systems. Systems that are nonlinear, nonequilibrium, deterministic, dynamic and incorporate randomness so as to be sensitive to initial conditions and have strange attractors, are said to be chaotic. Even deterministic equations may have unpredictable results and this is the essence of chaos. There is order in deterministic chaos and short-term predictions can be realized under certain circumstances though long-term predictions cannot be expected. Sensitivity to initial conditions is a vital factor and this can be highlighted by the following conjecture of Stephen Hawking: If the density of the universe one second after the big bang was greater by one part in a thousand billion (= one

followed by twelve zeros), the universe would have collapsed after ten years; had it been smaller by the same amount, the universe would have been substantially empty after 10 years. A critical density was necessary for this meeting to take place.

Butterfly effect: The flapping of the wings of a butterfly is a minute perturbation, but it may be adequate to change the weather in distant locations several days later. Often this is misrepresented as that if a butterfly flays its wings in Beijung, it will rain in New York in five days. An indication of the effect of nonlinearity.

Chaos theory in learning: Creative processes in both art and science often hinge on a transformation – a new idea – formed in the mind of an individual. These processes may be described as chaotic thinking; they involve a movement towards attractors and their basins. Tests under controlled laboratory conditions as well as uncontrolled naturalistic situations stress that creativity in learning is not merely a cognitive process but also a social practice in which attention shifts from the individual learner to learning as participation in society. Learning arises from participation based on negotiation and renegotiation of meaning in the world linked to the interaction of understanding and experience. Chaos theory can offer clues to creativity in learning even if the offer may seem peripheral to other approaches to learning.

The natural world and the brain are both inherently nonlinear and there is increasing awareness that chaos theory can be effectively useful for probing into developmental disabilities and the nature of human action itself. We must seek steps in the use of chaos theory to improve our understanding of the nature of learning and attitudes to learning. Learning process is analogous to motion in the physical sciences. A person or system is drawn from one position to another through a

process which can be precisely described by the modelling of the logistic equation which has three independent dynamical variables and hence makes possible the existence of chaos in the motion.

Can a deterministic problem embed randomness?

Consider the logistic equation (1) and take the parameter a = 4. $X_{n+1} = 4(x_n)\ (1-x_n)$ and the substitution $X_n = \sin^2 \pi\ \theta_n$ shows that $\pi\ \theta_{n+1} = 2\pi\theta_n + a$ multiple of π so that we have the simplified version

$$\theta_{n+1} = 2\ \theta_n \pmod 1 \qquad (2)$$

This transformation maps are closed interval [0,1] onto itself and if θ_0 is the initial state value by interaction, we see that $\theta_n = (2^n)\ \theta_0 \pmod 1$. θ_0 is a real number on the interval [0, 1] and has a unique binary representation which is in general infinite and equal to $b_1\ b_2\ b_3 \ldots\ldots$ in which every b_k is either zero or one. The interants $\theta_1, \theta_2, \theta_3, \ldots$ are obtained from this representation of θ_0 by shifting the binary point by one step, two steps, three steps, ... to the right and deleting the indices/ integers to the left of the binary point, thanks to the modulo 1 congruence in the formula (2) above. The solution of (2) certainly exists and is unique and for this reason, we have to reckon it as a deterministic equation. However, one can never construct these solutions over the entire interval [0, 1] in view of the nonavailability of a finite algorithm or a finite experiment for the construction of θ_n. The solutions are fully chaotic over the entire interval [0,1]. The digits $b_1, b_2, \ldots$ have the feature of randomness as commonly perceived in the sequential toss of a fair coin. The silvery lustre of determinism, existence and uniqueness of solutions and formal analytical representation of the solution is visibly tarnished, thanks to the prevalent chaos in the system.

We cannot find the θ_n- orbit here in full or in part unless the θ_n – orbit is given in full or in part before hand. We do not compute θ_n but do only copying or translation of the input into output. Chaos emerges here as a mystery, a Godel – type mystery, which only GOD can understand. In the context of the equation (2) chaos is deterministic randomness, deterministic because of the existence and uniqueness of the solution and randomness in view of the realization of the θ- orbit from a known random process. Historically, chaos conveys irregular, erratic, disorderly or seemingly unpredictable process, an antonym of order. It is an instance of exponentially sensitive dependence of the final state upon the initial state. This is alternatively expressed through the statements:

(i) the system has positive Lyapunov exponents,

(ii) the system has positive topological or metric entropy

(iii) the system has fractal attractors.

The message here is that we need to have an exponentially increasing knowledge of the present state to retain meaningful contact with a receding past or an advancing future.

Chaos, randomness, complexity: Consider an N digit binary sequence. The algorithmic complexity K_n of the digit equals the number of binary digits in the shortest computer program that prints the specified digit string and it is a quantitative measure of the informational compressibility or redundancy of the given chain. This measure K_n is essentially independent of the specific computer. Randomness is defined by the stipulation that $K_n \approx N$ and the normalized complexity of the sequence representing N is $K = \lim (K_n/N)$ = normalized complexity of the binary sequence for N. This limit exists in general. An infinite sequence with positive complexity is said to be random. Almost all binary strings have positive complexity and are therefore random. The digit strings

cannot be computed by a finite algorithm. If a finite sequence has its $K_n \approx N$, the sequence is random.

Almost all infinite real number digit strings are random but not a single one can be proved to be random. The difficulty lies in the noncomputability of K_n and this is not due to any theoretical mismanagement but is a consequence of innate human limitation. The difficulty lies in the self-reference that is inherent in the process. Difficult questions of self-reference are asked by systems that seek to gain information about themselves. A telling puzzle illustrates this feature. "A barber shaves everyone not shaving himself. Who shaves the barber?"

Order and chaos; Feigenbaum's Universality:

Chaos theory began towards the close of the last century with some ideas, concepts and results in the work of Henri Poincare' (1858-1912) on celestial mechanics. Natural laws of Newton and Kepler belong to the domain of order. Chaos was seen to be a condition in which nature fails to obey Laws. The observation that natural systems can easily switch from order to chaos or in the reverse direction was indeed a great puzzle. Fluid flow can change from laminarity to turbulence or vice versa. A regular heart beat may change into a fibrillating heart beat. We may see a change from predictability to unpredictability.

Intense study of the logistic equation (1), viz.,

$X_{n+1} = a\, x_n\, (1-x_n)$, $n = 0, 1, 2, \ldots$ (3)

shows that the two antagonistic states of order and chaos are governable by a single law and a great surprise is that there is a well defined route

which leads from one state to the other and that the route is universal. Abrupt qualitative changes – bifurcations – mark the transition from order into chaos and their universality means that such bifurcations are seen in diverse natural systems qualitatively and also quantitatively.

FIGURE 1

(Long term behaviour of the quadratic interactor (the logistic equation) for 'a' = 2). The first part shows the time series for a randomly chosen X_0 in the interval [0,1]. The orbit settles down to a fixed point – called the final state. Different initial values yield the same final state = 0.5. The second part shows the final state diagram, comprising just one point at height = 0.5.

Procedure for obtaining the long-term itinerant.

(i) select X_0 arbitrarily from [0,1] and carry out a large number of iterations, say 200 iterations. Drop the first 100 iterations and plot the rest to make the final state diagram. All these fall onto just one dot.

FIGURE – 2

(Final state diagram when a lies between 1 and 4)

For $a > 3$ the final state is not a mere point but a collection of 2, 4 or more points. For $a = 4$ we have chaos and the points of the final state densely fill up the complete interval (0, 1) on the vertical direction. This image is called the Feigenbaum diagram which is indeed a fractal closely related to the famous Mandelbrot set. The Feigenbaum diagram is an important icon of chaos theory and is a computer-generated image. The properties attached to the structure would never have been noticeable

without the computer. Modern chaos theory would not have been imaginable in the pre-computer era.

An essential structure in the Feigenbaum diagram (Figure 2) is that of a branching tree portraying the qualitative changes in the dynamical bahaviour of the iterator. Two branches bifurcate and on both these branches we have again bifurcation and then two branches bifurcate out of each of these. This is the scenario of the period – doubling regime.

When there is just one branch the long term behaviour of the system approaches a fixed final state, no matter what the initial state x_0 is. When these are two branches, the long term behaviour alternates between two distinct states, a lower one and an upper one. It is a case of periodic behaviour with period 2. The period of the final state behaviour increases from 2 to 4 and then to 8 and so on. This is the feature of period doubling arising from bifurcation at certain critical values of the control parameter (a). As a increases the Feigenbaum diagram structure has detailed and remarkable designs and chaos has set in. When $a = 4$, chaos governs the whole interval from 0 to 1.

FIGURE - 3

(Self – similarity in the Feigenbaum Diagram)

This figure comprises a sequence of close-ups. Start with Figure 2 ($1 < a \leq 4$) and in it introduce a rectangular window to the right of the first pitch-fork bifurcation. In the first close-up image magnify the rectangular window in the initial diagram of Figure 3 and show it upside down. This looks like the whole diagram. Again make a magnification of the rectangle and display the result upside down to obtain the second close-up. There is one more close-up in Figure 3, but this can be

repeated infinitely often. The final state diagram is a self-similar structure.

The branches in the period-doubling cascade become shorter as the control parameter a increases and there is a threshold value beyond which the branches of the tree cannot grow. This threshold value of a is s_∞ = 3.569945672 and is known as the Feigenbaum point, the value at which the sequence of rectangles in the close-up sequence of Figure 3 converge. The Feigenbaum point splits the final state diagram into two complementary parts. Onto its left there is the period-doubling tree and onto the right, the area corresponds to chaos. This is not an area of utter chaos and hides a variety of beautiful structures.

To quantify the way the period – doubling tree approaches the Feigenbaum point, Feigenbaum conducted many numerical experiments in and around 1975. If δ_k is the length in the direction of the a – axis between the K th bifurcation occurring for $a = a_k$ and $a = a_{k+1} > a_k$ (k = 1, 2, ...), $\delta_k = a_{k+1} - a_k > 0$ and the ratio $\delta_{k-1}/\delta_k = (a_k - a_{k-1})/(a_{k+1} - a_k)$ is approximately 4.669201.6091029... and numerical experiments with different interators confirmed that this constant was the same. This number δ is hence called the Feigenbaum constant and in view of its appearance in different systems like dripping faucets, oscillation of liquid helium and fluctuation of gypsy moth populations, it is called the universal constant of Feigenbaum.

FIGURE – 4

This universal constant δ is of utmost importance in the study of chaos theory just as π is in the geometry of circle and in the theory of numbers. It is a predictable constant in the world of chaos and in view

of its universality, it can be utilized to predict the bifurcations subsequent to two consecutive bifurcations. Experimental physicists discovered that the period – doubling scenario and the value of δ manifest themselves in real physical experiments like the dynamic behaviour in certain fluid flows apart from mathematical models.

FIGURE - 5

Figure 5 shows the final – state diagram for the logistic equation when the parameter '$\underline{a}$' lies between 2.8 and 4.

FIGURE - 6

FIGURE – 7

In the final-state diagram to the left of the Feigenbaum point s_∞ = 3.569945672 we have a self-similar fractal tree which describes the period – doubling scenario of the quadratic interator (1). There is a lead from the orderly behaviour of the dynamics right upto the beginning of the first pitch-fork bifurcation as the parameter ($\underline{a}$) increases from 1 to 3. This part (from a = 1 to a = 3) is a simple stable situation and the interation always leads to a rest point. Figure 7 shows the time series of the inial point (x_0) = 0.1 for $\underline{a}$ = 1.75 and the final state A (1.75) to which it settles down is 3/7. The approach from any initial value x_0 for this value of '$\underline{a}$' is the same final rest point (= 3/7). In the second picture of Figure 7 we see that A (2.75) = 7/11 but the approach to it passes through some oscillations around the final state before setting down to the rest position A (2.75). The dynamics in both the cases does allow the prediction of the result in the long run, no matter what the initial value is. The final rest point is thus an attractor of the iterations whatever may be the value of x_0. If x_o = 0 or 1, x_n = 0 for all n = 1,2, ... and the

points x = 0, x = 1 are fixed points of the transformation. The point $x=^{(a-1)}/_{a}$ is also a fixed point of the logistic equation and it is a stable fixed point if $1 < a < 3$, i.e. it is an attractor fixed point. These can be seen graphically by drawing the two curves: $y = ax(1-x)$ and $y = x$ over the range $0 \leq x \leq 1$ and allowing the full range of (a), viz., $0 < a \leq 4$.

The period – doubling cascade is universal for all mappings of the form $X_{n+1} = f(x_n)$ if the function f(x) has a single, locally quadratic, maximum. To state this precisely, the Schwarzian derivative of f(x) is to be negative in the bounded interval. The Schwarzian derivative of f(x) is

$$.f^{111}(x)/f^{1}(x) \; - 3/2 \; (f^{11}(x)/f^{1}(x))^2 \qquad (4)$$

For the quadratic iterator (1), the above is transparently negative since $f^{111}(x) = 0$.

For the quadratic iterator (1), the critical values of the parameter (a) at which the period – doubling bifurcations occur are given in the following table.

N	Cycle type	The value of 'a' at which the cycle appears.
1	2 – cycle	3.00
2	4 – cycle	3.44988
3	8 – cycle	3.54408
4	16 – cycle	3.56872
5	32 – cycle	3.5698912
.	.	.

.	.	.
∞	A periodic attraction	$3.569945672 = s_\infty$

Figure 8 shows the time profile of the logistic equation (1) for the parameter (a) = 2.75, taking $x_o = 0.1$. There are three intervals L_o, C, R_o on the a – axis which belong to three central mountains of the time profile. The pre image(s) of the initial end point of L_o(i.e. the pre images of α_1) are α_2, P_2; consideration of the pre images of α_2, then of α_3, ... shows that the valleys of the time profile are at the points $\{\alpha_k, P_k\}$. If the initial point $X_k = \alpha_1$ or β_1 the first k+1 interations are $\alpha_{k-1}, \alpha_{k-2}, \ldots, \alpha_1$, $^1/a$, $^{a-1}/_a$ and then onwards the $^{a-1}/_a$ is a rest point (a fixed point).

From order to chaos; the period doubling scenario:

Figure 9 shows the time series for a = 3.1 and the initial value 0.1. The iteration leads to a final state which consists of two points $x_1(a)$ and $x_2(a)$. Figure (10) shows that the left graph with $x_o = 0.075$ gives rise to the dynamics of the staircase and then the line spiralling in a clock-wise manner running into a repeating loop. The right graph with $x_o = 0.65$ has the orbit spiralling outwards towards the same loop as on the left graph. When a > 3, the fixed point $^{a-1}/_a$ is not an attractor but a repeller and close to $P_a = {}^{a-1}/_a$, iterations are pushed away. The fixed point $^{a-1}/_a = P_a$ loses its stability as (a) crosses the border $b_1 = 3$. This is the bifurcation point first noticed as the parameter (a) rises from the value 1 towards the final bound 4. The attractiveness to the state A(a) has been taken over by the loop which oscillates between $x_1(a)$ and $x_2(a)$. It is a 2-cycle or an orbit of period 2. x_1 is transformed to x_2 and x_2 to x_1.

Figure 11 gives the time profile for $\underline{a}$ = 3.1, $\underline{a}$ = 3.236 and $\underline{a}$ = 3.4. The similarity of the profile between $^1/_a$ and $^{a-1}/_a$ with the figure 8 is clear. If the logistic equation (1) is represented as $X_{n+1} = f_a(x_n)$, the orbit now belongs to the compounded iterator $f_a(f_a(x_n))$ which may be called the second iterate of $f_a(x_n)$.

$.f_a(x) = ax(1-x)$, $f_a f_a(x) = a[ax(1-x)][1-ax(1-x)] = -a^3x^4+2a^3x^3-(a^2+a^3)x^2+a^2x.$

This is a biquadratic and has four fixed points. $\{0, P_a, X_1(a), X_2(a)\}$ and these are the two fixed points of f_a and the two elements of the 2-cycle.

Figure 12 shows the period-doubling tree over the parameter range (b_1 to s_∞) and a close-up of it over (b_2 to s_∞). The relative positions of b_1, b_2, b_3 and b_4 show that $(b_3 - b_2)/(b_4 - b_3)$ is almost equal to $(b_2-b_1)/(b_3-b_2.)$

The distance $d_k = b_{k+1} - b_k$ (k = 1,2,...) between successive bifurcations decrease rapidly and the ratio $d_k/d_{k+1} = \delta_k$ has the limit

δ = 4.66920160910 29...

This number called Feigenbaum constant is the same for a wide range of different iterators and was discovered in October 1975.

Earlier in FIGURE 6 we have seen the period-doubling tree showing the final-state diagram for the parameter values 1 to s_∞, the latter being the Feigenbaum point. In FIGURE 13 we see the period-doubling tree with scaling factors d_1/d_2 = 4.669 and the ratio ν_1/ν_2 = 2.3. The leaves of the tree form a strictly self-similar cantor set and this happens right at the Feigenbaum point s_∞. The final-state diagram reaches a new stage more delicate than that for the parameter values less than s_∞. For all values of the parameter in the open interval $3 < \underline{a} < s_\infty$ we have stable periodic orbits as the final states.

When $\underline{a} = s_\infty$ the final state of the iterator comprises an infinitely long non-periodic orbit in a cantor set which gets arbitrarily close to every point of the cantor set. We are then at the entrance to chaos.

Self-similarity features: When (a) lies in the open interval $1 < \underline{a} < 3$, there is just one attractive fixed point at $P_a = {}^{a-1}/_{a}$ and all orbits having an initial value x_o with $0 < x_o < 1$, converge to that attraction. The way initial values are attracted changes at $a = s_1 = 2$ and this (s_1=2) is the super attractive case. For $\underline{a} < s_1$, initial values are attracted monotonically and when $\underline{a} > s_1$, there is an orbit spiraling about the fixed point. When $a = b_1 = 3$, the fixed point P_a is unstable and there is the birth of an attractive two-cycle. The old fixed point P_a continues to exist, but it repels. The two-cycle undergoes all the changes we have seen for the fixed point. The parameter value $a = s_2$ corresponds to the super attractive case. The two-cycle becomes unstable at $a = b_2$ and then we have a four-cycle. At each period-doubling bifurcation, the dynamics of the iteration becomes more complex.

$f_a(x) = ax(1-x)$, $f_a^2(x) = f_a(f_a(x))$, $f_a^3(x) = f_a(f_a^2(x))$ $s_1 = 2$, $s_2 \cong 3.236$, $s_3 \cong 3.498$, ... $s_\infty = 3.5699456$ ...

Chaos to order:

Consider the range $s_\infty < \underline{a} \leq 4$.

FIGURE - 14

The figure is the second part of the final state diagram of the logistic equation model (1). The picture is the chaotic mirror image of the period-doubling tree and the features of the period-doubling are seen now in reverse order. In the first part pertaining to the range $1 < a < s_\infty$, we can predict for each parameter value (a) what the dynamics is. Now

with $s_\infty < a \leq 4$, we cannot distinguish stable periodic behaviour from chaotic behaviour.

For $\underline{a} = 4$ there is only one band spanning the whole interval (0, 1). As $\underline{a}$ decreases this band slowly narrows. Figure 15 shows the value m_1 of $\underline{a}$ where the band splits into two parts. At $m_2 < m_1$, there is further splitting to a total of four parts. In the figure 15, there is a magnification of the final state diagram between s_∞ and m_1 at the window framed in black next to the Feigenbaum point. There are more band splitting parameter values. In fact, there is a decreasing sequence of parameter values ($m_1 > m_2 > m_3 > \ldots$) ($> s_\infty$) at which we observe splitting into 2, 4, 8, ... bands. This is another instance of the self-similarity of the final state diagram at the Feigenbaum point. This sequence decreases, passing in the limit to the Feigenbaum point. $m_\infty = s_\infty$. The distances of the band merging points are $e_k = 1m_{k+1} - m_k1$. As one may hopefully surmise, e_k/e_{k+1} converges to the constant δ =4.6692 .... (Feigenbaum constant).

Figure 16 shows in the top a typical time series of f_a for the parameter value a = 3.67 slightly below m_1. The behaviour is chaotic and there is oscillation from step to step back and forth between two distinct bands (levels). The dynamics of f_a^2 (lower figures) shows only chaotic movement either in the upper or the lower band. Band splitting is also a kind of period-doubling bifurcation.

In Figure 17 we see in the top the histogram for a = 4 and in the bottom we have the histogram for $a = m_1 = 3.6785$. The latter looks a bit like two differently scaled copies of the diagram for $\underline{a} = 4$ fitted together. This is an instance of the mixing behaviour of the quadratic iterator. The three characteristic features of chaos are (I) sensitive dependence on initial conditions; (ii) mixing behaviour and (iii) dense periodic points. All

these (mathematical) features are seen in the process of kneading of dough. The kneading process is closely related to the quadratic iterator (1).

In Figure 18 we have in the top the pictures of f_a and f_a^2 for $\underline{a} = 4$. The generic parabola given by $y = 4x(1-x)$ fits precisely into the square $[0,1] \times [0,1]$. In the bottom we have the picture for $\underline{a} = m_1$ and the chaos curve here is quartic but looks parabolic in the outlined region enclosed by a square of dots. We expect to see chaotic behaviour in the region which spans the interval $(1/a, a-1/a)$ and when the iterant gets into the region, it is trapped therein. This corresponds to the lower band in the final state diagram at $\underline{a} = m_1$. The upper band corresponds to the part of the graph of f_a^2 enclosed by the small dotted square. Here also the iteration is trapped and spans the interval $(a-1/a, a/4)$.

In Figure 19 we have the graphical iteration of a few initial values equally spaced near 0.5. These iterations for each of the chosen initial values show the graph drawn. The iteration never leaves the outlined square and the points of the final-state diagram lie between $v_a = f_a(0.5)$ and $f_a(v_a)$. For $a = 4$, $v_a = 1$ and all further iterates are zero. For $a = m_1$, $f_a^2(v_a) = P_a = a-1/a$ = the fixed point of f_a and all further iterates are the same.

In Figure 20 we see the close-up of the final-state diagram between $\underline{a} = 3.828$ and $\underline{a} = 3.829$ next to the line of critical values of $v_a = f_a(0.5)$ shown in dashed patterns. This close-up shows a small part of one of the white windows which interrupt the chaotic region of the final-state diagram. The number of such windows is infinite and the windows correspond to stable periodic cycles.

In Figure 21 we see the windows of period 3, period 5 and period 7. The bottom parts of the figure are two successive close-ups of the part marked by the black frame. There is self-similarity and we see smaller copies of the whole final-state diagram. There is again the scenario of period-doubling, chaos and band-splitting, though on a smaller scale. In stead of $f_a(x)$, it is $f_a^3(x)$ that can explain the features of the process. Period-doubling begins when the three fixed points of $f_a^3(x)$ loose their stability and six new fixed points of $f_a^6(x)$ are born. We have then a six-cycle for f_a. As $\underline{a}$ increases further, each of the stable fixed points of f_a^6 will undergo a period-doubling bifurcation. The relative length of the intervals for which these stable cycles exist, will be governed once again by the universal number $\delta = 4.669 \ldots$. At the end of this period-doubling scenario near $\underline{a} = 3.8415\ldots$ these will again be a transition to chaotic behaviour very much like that at the Feigenbaum point s_∞.

In Figure 22 we have the picture of the third iterate f_a^3. On the left is the super attractive case and at the centre in it is the segment looking like a small parabola. Changes of this small part are responsible for the complete scenario of period-doubling which ends at $\underline{a} \cong 3.857..$ This is seen in the middle picture of Figure 21. The corresponding picture or graph of f_a^3 is seen in the Figure 22 on the right side. We see a generic parabola at the centre.

The backdoor to chaos:

Periodic windows interrupt the chaotic region in an extreme way. At the beginning of such a window, there is a sudden change in the long-term behaviour of the iterator (1). Consider the value of $a = w_3 = 3.82843..$ at which the period – 3 window starts. For the parameter $\underline{a}$ slightly above w_3 there is a stable cycle of period 3. When the parameter $\underline{a}$ decreases to a value slightly below w_3, we stumble into chaos spanning the whole interval $[f_a(v_a)\ v_a]$. This is a different route to chaos unlike the

orderly path of periodic-doubling bifurcation. The chaos for a below w_3 manifests itself only in the long run and short term behaviour is indeed elusive, suggesting stable periodic orbits. In Figure 23 we see the time series of $x_o = 0.5$ for $a = 3.82812 < w_3$ and it is only after more than 50 iterations that the chaotic regime surfaces out. In the right picture we have the iteration f_a^3 with $x_o = 0.5$ and $a = 3.82812$. The time-axis is scaled up here by the factor 3 to facilitate better viewing and comparison.

The above numerical experiment is a new type of iterative behaviour, tensely described as intermittency. It is a turbulent 'burst' interrupting the otherwise periodic motion and is an important route to chaos different from the sequence of period doubling bifurcations.

Image encoding for the human brain:

The concept of iterated function systems (IFSs) is a useful factor for the construction of fractals and for better understanding of the number theoretical characterizations of some classical fractals like the Cantor ternary set and the Sierpinski gasket. Multiple reduction copy machine (MRCM) blue print is a set of transformations which describe the IFSs and provides an encoding of the image.

The visual functions of the human brain are comparable to the algorithmic designs of computers. Fractal geometry – which is the geometry of chaos – offers a powerful modelling framework for encoding general patterns and the human brain can distinguish encodings that correspond to different patterns in a large number of situations. One may therefore speculate that the human brain uses fractal-like encoding schemes.

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Legend of Figures.

1. Time series and Final State (a = 2)
2. Final-state Diagram ($1 < a < 4$)
3. Self-similarity in the Feigenbaum Diagram
4. The Feigenbaum constant and universality
5. Final-state Diagram ($2.8 < a < 4$)
6. The period-doubling tree ($1 < a < s_\infty$)
7. Stability of Final state (Time series)
8. Time profile (a = 2.75)
9. Stable oscillation (a = 3.1). The period-doubling scenario
10. Graphical iteration (a = 3.1)
11. Time profile (a = 3.1, a = 3.236, a = 3.4)
12. Period-doubling tree (Two close-ups)
13. Period-doubling tree: Entrance to chaos
14. Final-state Diagram: Part two (chaos to order)
15. Band splitting
16. Time series (a = 3.67)

17. Mixing Histograms ($a = m_1 = 3.6785$; $a = 4$)
18. .f_a and f_a^2 ($a = 4$,, $a = m_1$)
19. Initial values near 0.5
20. Start of the period-3 window.
21. Windows of periodicity
22. Third Iterate f_a^3
23. Intermittent time series